To Draw the Shortest Line Between Two Points on an Arbitrary Curved Surface

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* Let $AE = x$, $EB = y$, and from the point B is understood to be erected the line $Bb = z$, normal to the plane AEB and meeting the surface of the curve at b, and let there be given an arbitrary equation expressing the relationship of the three coordinates x, y, z, where the relationship is determined by the nature of the surface.

Now the line sought for will be lbc, from each of whose points l, b, c is understood to be dropped to the underlying plane AEB the normals lL,bB, cC, forming the projection [of the desired curve] LBC. Let the plane IBb, whose common section with the plane AEB, which is the line IB, touches therefore the projection LBC in the element BC, hence the plane IBb touches the sought-for curve in the element lbc.

Also let the plane bGI be conceived as tangent to the curved surface t b, and meeting the plane AEB in the common section GI, and [because each of the planes bGI and IBb touches the sought-after curve in the element bc], there will be extension of the same element bc, and will exactly connect with the curve in this element. Let BE be produced, so long as the segment IG runs to G, and from the point G let fall to BI the normal GH, which will then be perpendicular to the plane IBb. From the point H let the normal Hh be dropped to the line bI, and from the point b to the line cC the normal

*The Author communicated this Solution to Cl. KLINGENSTIERNA, Professor of Maths at the University of Upsala, before the end of the Year 1728; later the same KLINGENSTIERNA transcribed it to paper, so that it would be preserved.

bc. This being done, call

$$\begin{aligned} EF = BD &= dx, \\ DC &= dy, \\ BC = bc = \sqrt{x^2 + y^2} &= ds, \\ ce &= dz, \end{aligned}$$

touching below BG, which from the nature of the surface given in x, y, and z, $= T$.

The basis of the solution consists in this, that the plane crossing through three infinitely close points on the sought-for curve is orthogonal to the plane tangent to the curved surface. And so what is sought are two angles: one which the tangent plane bGI makes with the plane IBb , the other that the plane through three points of the desired curve makes with the same plane IBb. Having found these angles, it is determined that their sum must equal a right angle and the equation of the Problem will be considered solved.

Here is the procedure for this purpose. Because of the similarity of the triangles CBD,BGH,

$$CB\colon BD = BG\colon GH, \quad \text{or} \quad ds\colon dx = T\colon GH, \quad \text{whence} \quad GH = T\,dx\colon ds.$$

Because of the similarily of the triangles BCD, GBH,

$$BC\colon CD = GB\colon GH, \quad \text{or} \quad ds\colon dy = T\colon BH, \quad \text{whence} \quad BH = T\,dy\colon dx.$$

Because of the similarity of the triangles ceb, bBI,

$$ce\colon eb = bB\colon BI \qquad \text{or} \quad dz\colon ds = z\colon BI, \qquad \text{whence} \quad BI = z\,ds\colon dz.$$

From BI is subtracted BH, giving

$$IH = z\,ds\colon dz - T\,dy\colon ds.$$

Because of the similarity of the triangles bce, IHh,

$$bc\colon ce = IH\colon Hh, \qquad \text{or} \quad \sqrt{ds^2 + dz^2}\colon dz = \left(\frac{z\,ds}{dz} - \frac{T\,dy}{ds}\right)\colon Hh,$$

whence

$$Hh = \left(\frac{z\,ds}{dz} - \frac{T\,dy}{ds}\right) \times dz\colon \sqrt{ds^2 + dz^2} = \left(z\,ds^2 - T\,dy\,dz\right) : ds\sqrt{ds^2 + dz^2}.$$

Now because GH is normal to the plane IBb and Hh normal to the line bI, which is the section common to the planes IBb and bGI, the triangle HhG will be orthogonal to each of the planes IBb and bGI, and indeed, because of the right angle GHh, $hH\colon HG =$ radius: tangent of the angle of inclation HbG. And consequently , taking unity for the radius, the tangent of the angle of inclination HhG will be

$$\frac{HG}{Hb} = \frac{T\,dx\,\sqrt{ds^s + dz^2}}{z\,ds^2 - T\,dy\,dz}.$$

The other angle, which the plane passing through three infinitely close points of the desired curve lbc makes with the plane IBb, is now investigated.

In the projection LBC [Fig. 2, conceived as an infinitely small portion of the cylindroidal surface, flattened on the line bB, so that LB and BC are two elements of the curve LBC, which is the basis of the cylindroid] are three points L, B, C whose distances are equal and infinitely small; and there be the three correponding points l, b, c in the sought-for curve through which the said plane passes. Let LB be produced to ζ [Fig. 3] and lb to β, so that $B\zeta = BC$, and $b\beta = bc = bp$, and the remainder being made, as in the figure, and $LB = BC = ds$ being assumed constant, we shall have $fe = -d\,dz$, and because of the similiarity of the triangles bce, fcp,

$$bc\colon be = fc\colon cp, \qquad \text{or} \qquad \sqrt{ds^2 + dz^2}\colon ds = -d\,dz\colon cp,$$

$$\text{whence} \qquad cp = -ds\,ddz\colon \sqrt{ds^2 + dz^2}.$$

In order that $p\beta$, or that which ought to be considered equal to $C\zeta$, be discovered, let the projected curve be LBC, [Fig. 3] $BC = ds$, $BD = dx$, $CD = dy$, the tangent to B the line $B\zeta$, to which from the point C falls the normal $C\zeta$ and from the point ζ to CD the normal ζO.

Because of the similarity of the triangles BCD and ζCO,

$$BD\colon BC = CO\colon C\zeta, \quad \text{or} \quad dx\colon ds = d\,dy\colon C\zeta, \quad \text{whence} \quad C\zeta = ds\,ddy\colon dx = p\beta.$$

Now since cp and $c\beta$ are normal to bc, which is a common segment of the planes IBb or CBb, and of the plane passing through the three points l, b, c, the triangle $cp\beta$ will be in the plane normal to each of these planes, and because the angle $cp\beta$ is right, we shall have $cp\colon p\beta =$ radius: tangent of the angle of inclination $pc\beta$. So, taking unity for the radius, the tangent of the angle of inclination $pc\beta$ will be

$$\frac{p\beta}{cp} = \frac{ds\,ddy\colon dx}{-ds\,ddz\colon \sqrt{ds^2 + dz^2}} = \frac{-ddy\sqrt{ds^2 + dz^2}}{dx\,ddz}.$$

Because therefore, by the prior foundation of the Solution, the sum of the angle $pc\beta +$ and the angle HbG [Fig. 1] is a right angle; the product of the tangents of the angles will be = the square of the radii = 1. Hence, the tangent of the angle $pc\beta$, just now found, $-d\,dy\sqrt{ds^2+dx^2}\colon dx\,ddx$ in the tangent of the angle HhG, which was found above to be $T\,dx\sqrt{ds^2+dz^2}\colon (z\,ds^2 - T\,dy\,dx)$, is obtained the equation satisfying the problem:

$$\frac{T\,dx\sqrt{ds^2+dz^2}}{s\,ds^2 - T\,dy\,dz} \times \frac{-d\,dy\sqrt{ds^2+dz^2}}{dx\,ddz} = 1,$$

$$\text{or} \qquad (ds^2+dz^2)T\,ddy = (T\,dz\,dy - z\,ds^2)\,ddz.$$

Scholium I*

This is to be noted, which Cl. Klingenstierna disregarded, that the given curved surface can of course also be considered cut by the planes parallel to AE itself crossing through the point B and perpendicular to the plane AEB, which cuts make in the curved surface, other given curved lines whose subtangents to each point lookng B, are said to $= \theta$. Where changing T into θ, dy to dx, and ddy to ddx, will appear this other equation satisfying the Problem

$$(ds^2+dz^2)\theta\,ddx = (\theta dz\,dx - z\,ds^2)\;ddx;$$

or since $-dx\,ddx = dy\,ddy$, multiplying by $-T\,dx\colon dy$ will produce

$$(ds^2+dz^2)\,\theta\,T\,ddy = \left(-\theta\,T\,dz\,dx^2\colon dy + T z\,dx\,ds^2\colon dy\right)\;ddx.$$

On the other hand, as is easily demonstrated, it can generally be described for any desired curve on the surface, that

$$\frac{\theta\,dy + T\,dx}{\theta T} = \frac{dz}{z}, \qquad \text{or} \qquad \theta T\,dz = \theta z\,dy + T z\,dx,$$

where substituting of this value for $\theta T\,dz$, in the previously found equation, results in

$$(ds^2+dz^2)\theta T\,ddy = (-\theta z\,dx^2 - T z\,dx^3\colon dy + T z\,dx\,ds^2\colon dy)\,ddz$$

*Of these Notes, this and the following are the words of the Author himself and substituting dy^2 for $ds^2 - dx^2$,

$$(ds^2 + dz^2)\theta T\, ddy = (-\theta z\, dx^2 + T\, dx\, dy)z\, ddx.$$

then, having put again in the place of θT and z, their proportionals $\theta\, dy + T\, dx$ and dz, the equation is changed into

$$(ds^2 + dz^2) \times (\theta\, dy + T\, tx)\, ddy = (-\theta\, dx^2 T\, dy\, dx)\, dz\, ddz,$$

that is,

$$\frac{dz\, ddz}{ds^2 + dz^2} \quad = \quad \frac{\theta\, dy + T\, dx}{-\theta\, dx + T\, dy} \times \frac{ddy}{dx} \quad = \quad \frac{\theta\, ddx - T\, ddy}{\theta\, dx - T\, dy}.$$

Note. The same equation is also immmediately deduced from the first discovered

$$(ds^2 + dz^2)\, T\, ddy = (T\, dz\, dy - z\, ds^2)\, ddz.$$

Indeed, multiplying by θ, and afterwards substituting for $\theta T\, dx$ its value $\theta z\, dy + Tz\, dx$, derived from

$$(ds^2 + dx^2)\theta T\, ddy = (\theta z\, dy^2 + Tz\, dydx - \theta z\, ds^2)\, ddz$$

or, since $dy^2 - ds^2 = -dx^2$,

$$= (-\theta\, dx^2 + T\, dy\, dx)z\, ddx;$$

and I write $\theta\, dy + T\, dx$ and dz for θT and z, these being proportional; we have

$$(ds^2 + dz^2) \times (\theta\, dy + T\, dx)\, ddy = (-\theta\, dx^2 + T\, dy\, dx)\, dz\, ddz,$$

and thus, as before,

$$\frac{dz\, ddz}{ds^2 + dz^2} = \frac{\theta\, dy + T\, dx}{-\theta\, dx + T\, dy} \times \frac{ddy}{dx} = \frac{\theta\, ddx - T\, ddy}{\theta\, dx - T\, dy}.$$

For the three coordinates x, y, z we write t, x, y; this be the an equation expressing the nature of the curved surface, as did Cel. EULER, with $P\, dx =$

$Q\,dy + R\,ds$; (putting $ds = 0$) it becomes $P\,dx = Q\,dy$; and indeed, $P{:}\,Q = dy{:}\,dx = y{:}\,T$, where $T = Qy{:}\,P$; now taking $dx = 0$, it will be $Q\,dy = -R\,dt$, that is, $-R{:}\,Q = dy{:}\,dt = y{:}\,\theta$, thus $\theta = -Qy{:}\,R$. Therefore, in my formula,

$$\frac{dz\,ddz}{ds^2 + dz^2} = \frac{\theta\,ddx - T\,ddy}{\theta\,dx - T\,dy},$$

if for x, y, z, T, θ, are written respectively $t, x, y, Qy{:}\,P, Qy{:}\,R$, and for ds, which for me is supposed constant, is put $\sqrt{ds^2 + dx^2}$, this result is produced:

$$\frac{P\,ddt + R\,ddx}{P\,dt + R\,dx} = \frac{dy\,ddy}{dt^2 + dx^2 + dy^2}.$$

If in my earlier equation $(ds^2 + dz^2)T\,ddy = (T\,dz\,dy = z\,ds^2)\,ddz$ is applied to the Eulerian letters, and for T is written $Qy{:}\,P$, the somewhat simpler equation

$$\frac{Q\,ddx}{Q\,dx\,dy - P\,dt^2 - P\,dx^2}.$$

emerges.

In another note, the method shown above, $(ds^2 + dz^2)\theta\,ddx = (\theta\,dz\,dx - z\,ds^2)$, employed on the other hand by Euler, writing $-Qy{:}\,R$ for θ, results in an equation differing very slightly from the previous:

$$\frac{Q\,dds}{Q\,dt\,dy + R\,dt^2 + R\,dx^2} = \frac{ddy}{dt^2 + dx^2 + dy^2}$$

But if it is compared to the previous section of these two equations, the equation will return expressing the nature of the surface of the curve, $P\,dx = Q\,dy + R\,dt$, clearly necessary to happen; indeed, the two equations are equivalent.

Scholium II

The method, up to this point explained for solving the Problem of the Shortest Line drawn on a given surface, is also servicable of this method for the solution of other difficult problems of this kind, to which the common methods perhaps only with difficulty, or completely fail to reach. For example the following is proposed:

Problem: To draw on a given surface a curved line, so that at an arbitrary point, the osculating plane has a given inclination to the tangent plane of the surface at the given point. Here I call the *oscillating plane* that which crosses through three points, infinitely close to one another, on the desired curve.

SOLUTION: If the angle of inclination is entirely right, this Problem coincides with the preceeding: namely, it is the sought-for shortest curve itself, for which we gave the equation. It is truly now that oblique angle of the inclination, whose given tangent is $= n$. It follows from my Theorem in *Actis Lips.* 1722, published in the month of July, that if the tangents of two angles are a and b, the tangent of the sum of the two will be $= (a + b)(1 - ab)$, evidently having selected unity for the total sine, or the tangent of the semi-right angle.

Since, therefore, the our angle of inclination is known, in the preceding Solution, to depend on the two parts, of which the one for the tangent has

$$a = T\,dx\sqrt{ds^2 + dz^2} \colon (z\,ds - T\,dy\,dz);$$

the tangent of the other is certainly

$$b = -d\,dy\sqrt{ds^2 + dz^2} \colon dx\,ddz;$$

these being substituted into $(a+b) \colon (1-ab)$ and because it comes out equalling n itself, it will be reasoned that

$$n = \frac{\left(\dfrac{T\,dx\sqrt{ds^2 + dz^2}}{z\,ds^2 - T\,dy\,dx}\right) + \left(\dfrac{-ddy\sqrt{ds^2 + dz^2}}{dx\,ddz}\right)}{1 + \left(\dfrac{T\,ddy \times (ds^2 + dz^2)}{z\,ds^2\,ddz - T\,dy\,dz\,ddz}\right)},$$

or

$$\frac{T\,dx\sqrt{ds^2 + dz^2}}{z\,ds^2 - T\,dy\,dx} + \frac{-ddy\sqrt{ds^2 + dz^2}}{dx\,ddz} = \frac{nz\,ds^2\,ddz - nT\,dy\,dz\,ddz + nT\,ddy(ds^2 + dz^2)}{z\,ds^2\,ddz - T\,dy\,dz\,ddz}.$$

The former member being reduced to a common denominator, and each term of the latter multiplied by dx, so that these members have a common denominator, where then neglected, it results that

$$\big(T\,dx^2\,ddz - z\,ds^2\,ddy + T\,dy\,dz\,ddy\big) \times \sqrt{ds^2 + dz^2}$$
$$= nz\,ds^2\,dx\,ddz - nT\,dy\,dx\,dz\,ddz + nT\,dx\,ddy \times \big(ds^2 + dz^2\big)\,.$$

Any equation, if treated as was done above in the NOTE after SCHOLIUM I, will be (destroyed/elicited), the operation being duly carried through to the end, this other equation:

$$\big(\theta\,dx\,dy\,ddx + T\,dx^2\,ddz - \theta\,dx\,dz\,ddy + T\,dx\,dy\,ddy\big)\,\sqrt{ds^2 + dz^2)}$$
$$= n\theta\,dz\,ddz - nT\,dx\,dy\,dz\,ddz + (n\theta\,dy\,ddy + nT\,dx\,ddy)(ds^2 + dz^2).$$

Therefore either of these two equations satisfies the Problem.

COROLLARY

Given the special case, in which the equation found above,

$$(ds^2 + dz^2)T\,ddy = (T\,dz\,dy - z\,ds^2)\,ddz.$$

can be reduced to its principle distinguishing characteristics. If for example the given curved surface of its nature, as all of its sections, the plane is made normal to the segment AE, then the straight lines are parallel to the ordinate EB, the surface of whose kind can be called *cylindroidal*. In this case, the subtangent T avoids infinity; and to such a degree $z\,ds^2$ infinitely small, with regard to $T\,dz\,dy$. And so, with $z\,ds^2$ deleted, and the remaining terms of the equation divided by T, the result is

$$(ds^2 + dz^2)\,ddy = dz\,dy\,ddz,$$

where

$$ddy\colon dy = dz\,ddz\colon (ds^2 +\ dz^2);$$

and multiplying by 2,

$$2\,ddy\colon dy - 2\,dz\,ddz\colon (ds^2 + dz^2),$$

and having supposed of the integrals by logarithms $ln\,dy^2 - l(ds^2 + dz^2)$, and passage being made to the numbers, $n\,dy^2 = ds^2 + dz^2$. Because in this case z is given by x and constants, having set $dz = p\,dx$, understanding by p whatever quantity be given in x and constants, and for ds^2 is written its value $dx^2 + dy^2$, and having changed the equation found into this

$$n\,dy^2 = dx^2 + dy^2 + pp\,dx^2$$

or

$$(n-1)\,dy^2 = (1+pp)\,dx^2,$$

and extracting the square root,

$$dy\sqrt{n-1} = dx\sqrt{pp+1}.$$

Now it is manifest, the member of this last equation $dxpp + 1$ to designate the element of the arc of of the curve, whose coordinates are x and $\int p\,dx$, or z; it is of the generating curve of the Cylindroid. And so, if that arc is called A, it will be that $dy\sqrt{n-1} = dx\sqrt{pp+1} = dA$, and having obtained the integrals $y\sqrt{n-1} = A$; is is the ordinate of the projection to the arc of the generating curve of the Cylindroid, as 1 to $\sqrt{n-1}$, or in constant ratio, which from elsewhere is not difficult to deduce, and confirms the validity of the preceding Solution.

Scholium III

It is possible to arrive by a more elegant manner to our equation discovered above [in Scholium I],

$$\frac{dz\,ddz}{ds^2 + dz^2} = \frac{\theta\,ddx - T\,ddy}{\theta\,dx - T\,dy},$$

without the laborious computation of the angle of inclination HhG, which certainly the plane tangent to the surface curve bIG makes with the plane bBI. Towards this goal, using imagination, paying attention to both Figures 1 and 2, where the subject plane into which the curve of projection LBC is called *horizontal*; certainly the plane bIB touches sought-for curve in the element bc; in addition I shall name the continuation *vertical*; the plane containing the three points cBI, that is, call the plane of the triangle $c\beta b$, as before, the *osculating plane*. Now because the ocsculating plane should be perpendicular to the tangent plane [of the] curved surface, and because $c\beta$ is perpendicular to either of the two planes [with] common section $b\beta$, the line $c\beta$ will be perpendicular to the sample surface tangent plane. And so because if the line $c\beta$ is continued downard, until it meets the horizontal plane in the point P, from which towards the extremities of the subtangent G and M, are drawn lines PG and PM, are held two triangles PcG and PcM, right angled in internal point c; indeed, the plane McG is tangent to the surface, to which the normal is cP, and in fact

$$\text{ang.}PcG = \text{right} = \text{ang.}PcM; \quad \text{thus} \quad PG^2 - cG^2 = Pc^2 = PM^2 - cM^2.$$

It is true that

$$cG^2 = CG^2 + cC^2 \quad \text{and} \quad cM^2 = CM^2 + cC^2,$$

which values being substituted and deleting the common factor Cc^2,

$$PG^2 - CG^2 = PM^2 - CM^2.$$

Now having considered that PR is itself parallel to MC, or perpendicular to GC produced, it is revealed that (of course the angle GCM is right), and PS is itself parallel to GC, or perpendicular to MC; certainly

$$PG^2 - CG^2 = PC^2 + 2CG \cdot CR, \quad \text{and} \quad PM^2 - CM^2 - PC^2 = 2CM \cdot CS,$$

whence

$$CG \cdot CR = CM \cdot CS, \quad \text{or} \quad CS : CR = CG : CM,$$

that is, the sides of the rectangular parallelogram RS are themselves reciprocally proportional to the subtangents. From this it follows that the tangent of the angle RCP is

$$\frac{RP}{RC} = \frac{CS}{CR} = \frac{CG}{CM} = \frac{T}{\theta}.$$

It is understood furthermore in Fig. 2, that the smaller side pc in the vertical plane is continued downward into the horizontal plane which occurs at the point V. [Fig. 4]. Let the lines cV and CV be drawn. First, cVP is a right triangle at V and similar to $cp\beta$ itself, because both PV and $p\beta$ are horizontal, lying in the common plane $cp\beta$, and also parallel; in fact, the plane cVP is nothing more than a continuation of the plane $cp\beta$. Second, CV is a continuation of the element BC, or a tangent of the projection LBC of the curve. And therefore, in order that the angle RCV be found, this is [Fig. 1] the angle BCD, whose tangent $= \frac{BD}{DC} = \frac{dx}{dy}$; thus appears. The angle CVP is right; the proof of this is easily imagined; and with cVP being also right, it will be, on account of the common side PV, that the tangent of the angle VcP to the tangent VCP as VC to Vc, which equals (since the right triangle cCV is similar to the right triangle bec in Fig. 2) $ec\colon bc = dz\colon \sqrt{ds^2 + dz^2}$. And so, because the tangent of the angle VcP, or

$$pc\beta = \frac{p\beta}{pc}$$

which equals, as found above [in Corollary I to Scholium I]

$$-ddy\sqrt{ds^2 + dz^2}\colon dx\, ddz.$$

Now certainly from knowledge of the tangent of the two angles RCP and VCP , by the help of my theorem presented in *Act. Lips. 1722*, the tangent of the angle composed from those RCV, or from that which is opposed to the vertex BCD, each has tangent $\frac{dx}{dy}$, certainly the tangent of the angle RCV will be

$$\frac{T}{\theta} + \frac{-ddy(ds^2 + dz^2)}{dz\, dx\, ddz}\colon \left(1 + \frac{T\, ddy(ds^2 + dz^2)}{\theta\, dz\, dx\, ddz}\right)$$

or

$$\frac{T\, dz\, dx\, ddz - \theta\, ddy(ds^2 + dz^2)}{\theta\, dz\, dx\, ddz + T\, ddy(ds^2 + dz^2)},$$

which reduces to

$$\frac{dz\, ddz}{ds^2 + dz^2} = \frac{\theta\, dy + T\, dx}{-\theta\, dz + T\, dy} \times \frac{ddy}{dx},$$

which, since $\frac{ddy}{dx} = -\frac{ddx}{dy}$, equals

$$\frac{\theta\, ddx - T\, ddy}{\theta\, dx - T\, dy},$$

as was found by the first method. [In SCHOLIUM I, before the NOTE.]

COROLL. I

In the case of the Cylindroid, where one or the other of the subtangents, e.g., CM, becomes infinite, CR and RP will have an infinite ratio; that is, the angle RCP vanishes, from where it is made of such size

$$\frac{dx}{dy} = \text{tangent of angle } VCP = \frac{-ddy(ds^2 + dz^2)}{dz\, dx\, ddz},$$

which immediately gives

$$\frac{dz\, ddz}{ds^2 + dz^2} = \frac{-dy\, ddy}{dx^2} = \frac{ddx}{dx},$$

which is the same equation which was considered in the above Corollary, except where that was x this is y. The rest is completed as in that place.

COROLL. II

In the case of the Conoid, where Cc is parallel to the axis, through which and through the axis itself crosses the plane cCG, but the planes parallel to the base form circles on the curved surface, so that once again the subtangents CM avoid infinity. And also by this way is made [see Fig. 5] $\frac{BD}{CD}$ = tangent of the angle VCP. That is true here[$-dx$ is selected, because, if x and y be increasing, it is assumed that z is decreasing.]

$$ec\colon bc = -dz\colon \sqrt{s^2 + z^2} = \tan VcP : \tan VCP.$$

But the tangent of angle VcP, or $pc\beta$, is

$$\frac{p\beta}{pc} = \frac{(y\, ddx + 2\, dy\, dx) \times \sqrt{ds^2 + dz^2}}{dy\, dz\, ddz}$$

so that the lower is pointed out

Hence the tangent of angle VCP equals

$$\frac{(y\,ddx + 2\,dy\,dx) \times \sqrt{ds^2 + dz^2}}{dy\,dz\,ddz} = \frac{BD}{CD} = \frac{y\,dx}{dy},$$

and from its reduction emerges

$$\frac{dz\,ddz}{ds^2 + dz^2} = \frac{y\,ddx + 2\,dy\,dx}{y\,dx}.$$

Integrating with logarithms, and then transiting to numbers, gives $yy\,dx = b\sqrt{ds^2 + dy^2}$. Truly, in order that this also not be made a laborious work, by the discovery of the angle of the surface tangent plane and the vertical plane.

Application of this method to the Conoidal Surfaces; or such that, by an arbitrary change to the curve of whatever any given position around the given axis is produced.

Let the vertex of the Conoid be a; the sought-for curve be lbc. From the vertex a, and from each of the points l,b,c, etc., of the curve, are understood to fall to the underlying plane normal to the axis aA, perpendiculars aA, lL, bB, cC, etc. forming [Fig. 5] the projection of the vertex A and the projection LBC of the desired curve. By the center A, and an arbitrary radius $AK = 1$, are described in the said plane to the axis aA of the Conoid, a true circle KEF , and lines ABE, ACF are drawn, meeting the projection LBC in the infinitely close points B,C, and the circumference KEF in E, F. Furthermore, let the plane IBb be conceived, whose common section with the plane BAK is the line IB, to touch the projection LBC in the element BC; whereby the plane IBb itself grazes the desired curve in the element bc. Furthermore, let the plane bGI be conceived as tangent to the Conoid in b, and meeting the plane BAK in the common section GI, and the plane IBb in the common section bI, which because whichever of the planes bGI and IBb meets the desired curve in the element bc, there will be produced of the same element bc, and indeed touches the curve in this element. Let AE be produced until the section IG meets in G; and the angle AGI will be right. From the point G let the normal GH be dropped to BI, which then will be perpendicular to the plane IBb; from the point H let the normal Hb be droppd to bI, and joined with Gb, and finally from the point c let the normal ce be dropped to bB. These being done, and calling the arc $KE = x$, $EF = dx$, $AB = y$, $DC = dy$, $BD = y\,dx$, $BC = \sqrt{dy^2 + yy\,dx^2} = ds$, $Bb = z$, $bc = -dz$, $T =$ the subtangent BG from the nature of the Conoid. Because of the similarity of triangles CDB,BHG,

$$CB\colon BD = ds\colon y\,dx = BG\colon GH = T\colon GH, \quad \text{whence} \quad GH = Ty\,dx\colon ds.$$

Because of the similarity of triangles CDB,IHG,

$$CD\colon BD = dy\colon y\ dx = GH\colon HI = \left(\frac{Ty\,dx}{ds}\right)\colon HI, \quad \text{whence} \quad HI = Tyy\,dx^2\colon ds\,dy.$$

Because of the similarity of triangles bec,HhI,

$$bc\colon be = \sqrt{ds^2 + dz^2}\colon (-dz) = HI\colon Hh = \left(\frac{Tyy\,dx^2}{ds\,dy}\right)\colon Hh,\ \text{whence}$$
$$Hh = Tyy\,dx^2\,dz :\ ds\,dy\,\sqrt{ds^2 + dz^2}.$$

Now, because GH is normal to the plane IBb, and Hb normal to the line bI, which is the common section of the planes IBb and bGI, the triangle HbG will be in a plane orthogonal to either of the planes IBb and bGI, and indeed because the angle GHb is right, $bH\colon HG =$ radius: tangent of the angle HhG of inclination. And so, assuming unity for the radius, the tangent of the angle of inclination HhG will be

$$= \frac{HG}{Hh} = \frac{Ty\,dx\colon\ ds}{-Tyy\,dx^2\,dz\colon\ ds\,dy\sqrt{ds^2 + dz^2}} = \frac{-dy\sqrt{ds^2 + dz^2}}{y\,dx\,dz}.$$

Another angle, which the plane through the three infinitely close points lbc of the desired curve makes with the plane IBb, is thus to be investigated. [See Figs. 6 and 7, which are derived from the image of figure 2, an infinitely small part of the surface of the cylindroid.]

In the projection LBC, there are three points L,B,C, of whose distances LB, LC, are equal and infinitely small; and there are in the desired curve lbc three corresponding points l,b,c, through which passes the said plane. Let LB be produced to ζ, and lb to β, so that $B\zeta = BC$, and $b\beta = bc = bp$. In order that the rest be in the Figure, and set $LB = BC = ds$ constant; then $lg = bm = -dz$, and $me = fc = -ddz$.

On account of the similitary of the triangles fmb, cpf,

$$bf\colon fm = fc\colon cp \qquad \text{or} \qquad \sqrt{ds^2 + dz}\colon ds = -ddz\colon cp,$$

whence

$$cp = -ds\,ddz\colon \sqrt{ds^2 + dz}.$$

In order that $p\beta$ be found, or that which should be thought equal, $C\zeta$, be in Fig. 7, the projected curve LBC, the center A, the tangent to C line CP and the tangent to B line BO, in which are falling from the center A the perpendiculars AP and AO, and in OB is produced the normal $C\zeta$.

On account of the similitary of the triangles BCD,BAO,

$$CB\colon BD = BA\colon AO, \qquad \text{or} \qquad ds\colon y\,dx = y\colon AO,$$

whence

$$AO = yy\,dx\colon ds,$$

and supposing a difference, setting ds constant

$$dAO = oP = (yy\,ddx + 2y\,dx\,dy) : ds.$$

On account of the similitary of the triangles BCD,BAO,

$$BC\colon CD = AB\colon BO, \qquad \text{or} \qquad ds\colon dy = y\colon BO,$$

whence

$$BO = y\,dy\colon ds.$$

On account of the similitary of the triangles BoP,$BC\beta$, Bo or

$$\beta o\colon oP - BC\colon C\zeta, \qquad \text{or} \qquad \left(\frac{y\,dy}{ds}\right) : \left(\frac{yy\,ddx + 2y\,dx\,dy}{dx}\right) = ds\colon C\zeta,$$

and thus

$$C\zeta = p\beta = (y\,ds\,ddx + 2\,dx\,dy) : dy.$$

Now because cp and $c\beta$ [Fig. 6] are normal to bc, which is the common line of the planes IBb, or CBb, and the plane through the three points l, b, and c, of crossed over, the triangle $cp\beta$ in the plane orthogonal to either of those planes will be right, and since the angle $cp\beta$ is right, cp: $p\beta$ equals radius: tangent of the angle $pc\beta$ of inclination. And supposing unity for the radius, the tangent of the angle of inclination $pc\beta$ will be

$$\frac{p\beta}{cp} = \frac{(y\,ds\,dds + 2\,dx\,dy\,ds) : dy}{-ds\,ddz\sqrt{ds^2 + dz^2}} = \Big(y\,ddx + 2\,dy\,dx)\sqrt{ds^2 + dz^2}\Big) : -dy\,ddx.$$

And so, because angle $pc\beta +$ angle HhG is right, the product of the tangents = the square of the radius = 1; whence the tangent drawn by the method shown the angles $pc\beta$, $(y\,ddx + 2\,dy\,dx)\sqrt{ds^2 + dz^2}$: $dy\,ddz$, into the tangent of the angle HhG, which above was shown to be $-dy\sqrt{ds^2 + dz^2}$: $y\,dx\,dz$, is obtained the equation

$$(y\,ddx + 2\,dy\,dx) \cdot \big(ds^2 + dz^2\big) : y\,dx\,dz\,ddz = 1,$$

or

$$\frac{yy\,ddx + 2y\,dy\,dx}{yy\,dx} = \frac{1}{2} \times \frac{2\,dz\,ddz}{ds^2 + dz^2}.$$

Taking integrations by logarithms,

$$\log yy\,dx = lb + \log\sqrt{ds^2 + dz^2},$$

and making the change back to numbers,

$$yy\,dx = b\sqrt{ds^2 + dz^2};$$

this equation exprsses the nature of the projection LBC.

Because z, by the nature of the Conoid is given in y and constants, it is put $dz = p\,dy$, understanding by p any function whatever of y itself. In the equation found, $yy\,dx = b\sqrt{ds^2 + dz^2}$, if for dz is written $p\,dy$, and for ds^2 is substituted its value $dy^2 + yy\,dx^2$, which being done, will have

$$\boxed{dx = \frac{b\,dy}{y}\sqrt{\frac{1+pp}{yy-bb}};}$$

in which equation, because the distinct parts are not limited, it is possible to have constructed the curve of the projection LBC by quadratures, this completed, if from the single points L,B,C, etc., are erected normals Ll,Bb,Cc,etc., the surface of the given Conoid meeting in the points l,b,c, etc., will be designated in the said sought-for surface curve lbc. *Q.E.F.*

First Example

If the given Conoid were the surface of a plane, parallel to the plane of projection ABK, it would contain z, and indeed $dz = p\,dy = 0$; in consequence, $p = 0$, and the general equation $dx = \frac{b\,dy}{y}\sqrt{\frac{1+pp}{yy-bb}}$ is changed into this, $dx = \frac{b\,dy}{y\sqrt{yy-bb}}$, or, in multiplying by b, $b\,dx = \frac{bb\,dy}{\sqrt{yy-bb}}$. The latter member $\frac{bb\,dy}{\sqrt{yy-bb}}$ is an element of a circular arc, whose radius $= b$ and secant $= y$; if this arc is called A, [having set the radius AK (Fig. 5) , which is arbitrary and assumed to be unity], $x = A$. Therefore, because y is always secant of the angle x or A, it is well known that the sought-for line is right, any circle whose radius is b, touches. If the given Conoid is a Cone, whose axis is to the base radius as n is to 1, so that $x = ny$, $dx = p\,dy =, dy$, and indeed $p = n$, and the equation to the projection

$$dx = \frac{b\,dy\sqrt{1+nn}}{y\sqrt{yy-bb}}, \text{ or } \frac{b\,dx}{\sqrt{1+nn}} = \frac{bb\,dy}{y\sqrt{yy-bb}}$$

and having obtained the integrals,

$$\frac{bx}{\sqrt{1+nn}} = \int \frac{bb\,dy}{y\sqrt{yy-bb}},$$

which is the arc of a circle with radius b and secant y. Which if called A, then $bx\colon \sqrt{1+nn} = A$, or (once more setting $AK = 1 = b$), $x\colon \sqrt{1+nn} = A$. Thus this equation can be constructed.

At the center A (Fig. 8), the circle LEF is described by the radius $AL = b$, which the line LD touches in L. From the center A is drawn out the secant AED, which meets the circumference LEF in E, and if the tangent LD in D, the angle LAF to the angle LAE is taken as $\sqrt{1+nn}$ to 1, and AF is produced as far as needed in B. Given the center A, the radius AD is described the arc of the circle DB orthogonal to AFB, occurring in B. This being done, point B will be to the desired projection LBC.

Corollary

If $\sqrt{1+nn}$ be a rational number, the curve LBC of the projection will be algebraic. Because it happens, as often as r be supposed be any number, whole or fraction, we shalll have $n = (1-rr)\colon \pm 2r$, or $n = \pm 2r\colon (1-rr)$; such a procedure is well known, put in place by the *Diophantean* method $\sqrt{1+nn} = r+n$, and $\sqrt{1+nn} = 1+rn$.

Example III

$dx = \frac{b\,dy}{y}\sqrt{\frac{1+pp}{yy-bb}}$

Let the given Conoid be a Sphere, whose radius $= a$, center the point A. By the nature of a Sphere

$$z = \sqrt{aa-yy}, \qquad dz = -y\,dy\colon \sqrt{aa-yy}, \quad \text{and} \quad p = -y\colon \sqrt{aa-yy}.$$

Substituting this value into the general equation, the latter is changed into*

$$dx = ab\,dy\colon y\sqrt{yy-bb}\cdot\sqrt{aa-yy}.\ ^*$$

*From this equation it can immediately be deduced (the expression above being considered the most complex which supports this) that the sought-for shortest curve on the surface of the sphere is a great circle. Indeed, because the latter part $ab\,dy\colon y\sqrt{y^2-b^2}\cdot\sqrt{a^2-y^2}$ is proportional to the differential of the arc of the great circle, whose sine is $n\sqrt{a^2-y^2}\colon y$ (where by n I mean $ab\colon\sqrt{a^2-b^2}$), as is certainly accessible to anyone who wishes to prove this, and because the quantity $n\sqrt{a^2-y^2}\colon y$ is itself proportional to the tangent of the meridian arc cut off between the points b and E, the shortest curve *cbl* on the surface of the sphere must be of the same nature, so that the sine of the arc KE is to tangent of the arc *Eb* in constant ratio. Moreover it is established in Spherical Trigonometry that this coincides with an arbitrary great circle, which in the point K obliquely cuts the circle KE which is assumed for the base. Indeed, it is everywhere, as the total sine is to the oblique tangent, so the sine of the indeterminate arc KE is to the tangent of the corresponding arc *Eb*. Therefore the shortest curve on the surface of the sphere is a great circle arc.

In order that the latter member $ab\,dy\colon y\sqrt{yy-bb}\cdot\sqrt{aa-yy}$ be reduced to a simpler form, we take $y=ab\colon u$, so that $dy=-ab\,du\colon uu$, and it is found that, the substitution being made,

$$-u\,du\colon\sqrt{aa-yy}=-n\,du\colon\sqrt{aa-uu}\cdot uu-bb.$$

Furthermore, having put $uu=cv$, and $2u\,du=c\,dv$, which being substituted, results in

$$-u\,du\colon\sqrt{aa-uu}\cdot\sqrt{uu-bb}=-\frac{1}{2}\,dv\colon\sqrt{-aabb\colon cc+(aa+bb)v\colon c-vv};$$

in addition, make $-v_(aa_bb)\colon 2c-t$, and $-dv=dt$, and the calculation being carried out results in

$$-\frac{1}{2}\,dv\sqrt{-aabb\colon cc+(aa+bb)v\colon c-vv}=-frac12\,dt\colon\sqrt{(aa-bb)^2\colon 4cc-t},$$

and multiplying either member by $(aa-bb)\colon c$,

$$(aa-bb)\,dx\colon c=(aa-bb)\,dt\colon 2c\sqrt{(aa-bb)^2\colon 4cc-tt},$$

and obtaining the integrals,

$$(aa-bb)x\colon c=\int\frac{aa-bb}{2c}\,dt\colon\sqrt{\left(\frac{aa-bb}{2c}\right)^2-tt}$$

which is the arc of the circle whose radius is $(aa-bb)\colon 2c$ and right sine is t, which arc if called A, will have $(aa-bb)x\colon 2c=A$, or (taking the arbitrary radius $AK[1]=$ radius $(aa-bb)\colon 2c$), $2x=A$.

The curve of the desired projection leads to the conducting of this calculation by this method: By the center A and radius $AK=(aa=bb)\colon 2c$, the circle KEF is described, and led out until it reaches the line AE; the arc $KF=2KE$, and FG will at the point F be normal to the extended radius $AK=t$. From here, if in AE, if the work be produced, it is taken that

$$AB=y=\text{ (by construction) }ab\colon u=ab\colon\sqrt{cv}=ab\colon\sqrt{\frac{1}{2}aa+\frac{1}{2}bb-ct},$$

so that the point B will be in the curve of the desired projection. But with this curve being algebraic, just as is well known from the construction, it is appropriate to investigate the algebraic equation between the rectangular coordinates AH,HB. It is asserted/named to this end that $AH = p$, $HB = q$. Because the angle FAK is bisected by the line ANE,

$$FA + AG \colon FG = AG \colon GN,$$

or

$$\left[(aa - bb) : 2c + \sqrt{(aa - bb)^2 : 4a - tt}\right] : t = \left[\sqrt{(aa - bb)^2 : 4c - tt}\right] : GN,$$

where

$$GN = t\sqrt{(aa - bb)^2 : 4cc - tt}.$$

Because of the similarity of the triangles AGN,AHB,

$$AG \colon GN = AH \colon HB,$$

or

$$\sqrt{(aa - bb)^2 : 4c - tt} \colon t\sqrt{(aa - bb)^2 : 4cc - tt} \colon \left[(aa - bb) : 2c + \sqrt{(aa - bb)^2 : 4cc - tt}\right] = p \colon q,$$

where they are considered to be mediums and extremes to each other; and dividing each side by

$$\sqrt{(aa - bb)^2 : 4cc - tt},$$

we shall have

$$q = pt \colon \left[(aa - bb) : 2c + \sqrt{(aa - bb)^2 : 4c - tt}\right];$$

hence

$$q\sqrt{(aa - bb)^2 : 4cc + tt} = pt - aaq \colon 2c + bbq \colon 2c.$$

If each member of this equation is raised to the square, it is found that, removing that which is demolished,

$$ct - (aa - bb)pq\colon (pp + qq).$$

Because the triangle AHB is right, we shall have

$$AB^2 = AH^2 + HB^2,$$

or

$$yy = aa\,bb\colon (\frac{1}{2}aa + \frac{1}{2}bb - ct) = pp + qq,$$

which being reduced gives

$$ct = \frac{1}{2}aa + \frac{1}{2}bb - aa\,bb\colon (pp + qq),$$

or

$$pp + qq = 2\,((aa - bb)pq + aa\,bb) : (aa + bb),$$

which equation pertains to the Ellipse DBE, whose major semiaxis $AD = a$, minor $AD = b$, rectangular coordinates $AH = p$, $HB = q$, showing that the angle DAH is semi-right.

For sending down BC normal to AD, and calling $AC = x$, $BC = y$, we shall have $x + y = q\sqrt{2}$, and $x - y = p\sqrt{2}$; and removing q and p from the equation

$$pp + qq = 2\,((aa - bb)pq + aa\,bb) : (aa + bb),$$

we shall have

$$aa\,yy + bb\,xx = aa\,bb.$$

Note

The general equation $dx = \frac{b\,dy}{y}\sqrt{\frac{1+pp}{yy-bb}}$ proves to be consistent with the Solution of the Celebrated Jac. Bernoulli in *Act. Erud. Lips.*,, year 1698, published on page 227. For if in his formulat $\int at\,dx\colon xx\sqrt{xx - aa}$, y is written for x, and b for a, the result is $\int bt\,dy\colon yy\sqrt{yy-bb}$, where for t is written its value $y\sqrt{dy^2+dz^2}\colon dy$, and for dz, as above, $p\,dy$, we shall have $\int \frac{b\,dy}{y}\sqrt{\frac{1+pp}{yy-bb}}$.

But the Worthy Gentleman deduced this solution with no uncertain foundation. For if he had extended the method to every kind of curved surface, certainly he would have given a general solution to the Problem posed by his Brother. The following method is generally extended with difficulty to the conceived problem.

Give the Conoid $ACEBDA$, whose axis AB be right to the circular base ECD, let the sought-for line be $KPIG$. The surface $ACEA$ is divided into infinite sectors, prolonged to the meridians APN, AIf, AGF, etc., infinitely close, and by the points P,I,G, etc., where these meridinans cut the curve $KPIG$, extending tangents PL, IL, GL, etc., cutting the axis AB in the points L,L,L, etc., separated from each other by an infinitely small distance; whence any two points L, L, where two very close tangents, e.g. GL and IL, cut the axis, can be considered as one and the same, and consequently the figure $LGIL$ can be considered as a triangle, and similarly is understood the remainder $LIPL$.

The plane of the triangle GLI is conceived as the axis near IL slighly lifted up, until with the following triangle ILP it constitutes one same plane GLP. Similarly, this plane GLP is conceived as near the line PL, until it exists in the same palne with the following triangle, and this

is continued along until $LGIPKL$ be reduced into a plane. (See Fig. 12.) Thus, this being done, it is manifest that the line $KOPIG$ reduced into the plane will obviously be the shortest straight line in the plane. Thus by nature of a straight line, the angle GLI is the equal of the different angles LIK, LGK; the angle $ILP = LPK - LIK$, etc.; that is, the angle which the nearest touch, v.g., GL and IL, include in a point L of the axis, where are thought to coincide, is equal of the different angles LGI or LIP, which touching LG or LI make with the curve $KPIG$ in G or I.

From this foundation the nature of the curve $KPIG$ will be investigated. Through whatever point on the supposed curve G (Fig. 11) is drawn part of the circle of parallel Gg, intercepted by medians AGF and AIF. From the points G and g are drawn to the axis AB normals GH, gH, and from the points F,f to the center of the base B the lines FB, fB. Assuming the radius of the base $BC = BF = 1$, setting $GH = x$, whose element $fF = dx$, a small arc of the meridian $Ig = ds$, and the tangent GL or IL=t.

Because of the similarity of the sectors BFf, HGg, $BF\colon HG = Ff\colon Gg$, or $1\colon x = dx\colon Gg$. Therefore $Gg = x\,dx$. Hence the angle $GLI = Gg\colon GL = x\,dx\colon t$.

The complement of the angle LIG in the semicircle is the angle GIg, whose tangent is to the radius as Gg to gI; wherefore, setting the radius $= BC = 1$, it will be the tangent of the angle GIg, or, by only a change of sign, the tangent of the angle $LGI = -Gg\colon gI - x\,dx\colon ds$. Therefore

$$\text{an element of the angle } LGI = -d\left(\frac{x\,dz}{ds}\right)\colon\left(1 + \frac{xx\,dz^2}{ds^2}\right).$$

But by the foundation provided above, the angle GLI is an element of the angle LGI; therefore is had the equation

$$\frac{x\,dz}{t} = -d\left(\frac{x\,dz}{ds}\right)\colon\left(1 + \frac{xx\,dz^2}{ds^2}\right);$$

or because $x\colon t = dx\colon ds$,

$$\frac{dx\,dz}{ds} = -d\left(\frac{x\,dz}{ds}\right)\colon\left(1 + \frac{xx\,dz^2}{ds^2}\right);$$

By integrating his equation, it is put that $x\,dz\colon ds = v$, and the substitution having been made,

$$v\,dx\colon x = -dv\colon (v + v^3) = -dv\colon (1 + vv),$$

or

$$-dx\colon x = dv\colon (v + v^3) = dv + 3vv\,dv)\colon (v + v^3) - 3v\,dv\colon (1 + vv);$$

and supposing integration by Logarithms,

$$l(d\colon x) = l(v + v^3) - \frac{3}{2}l(1 + vv) = lv - \frac{1}{2}l(1 + vv),$$

where

$$a\colon x = v\sqrt{1 + vv}, \text{ and } a\colon \sqrt{xx - aa} = v = x\,dx\colon ds;$$

and finally

$$dx = a\,ds\colon x\sqrt{xx - aa}.$$

Same Problem

To draw the shortest line on a given arbitrary curved surface.
By the method of maxima and minima

Solution. Having conceived on the horizontal plane AEC, from each of the points a, b, and c, on the desired curve, to have let fall perpendiculars aA, bB, cC, etc., which form the curve of the projection, any two contiguous elements of which are represented by AB, BC. Through A and B are drawn AE, BD, as elements of the abscissae, and EB, DC as elements of the projection of the curve near to the leading perpendiculars. Again, the point B in the prolonged element EB is understood to fall towards the very near location ζ, which corresponds to the point β on the curved surface; but itself cut by the vertical plane crossing throught the line $L'BF$, that cut forming a curve, which, because of the given surface, and itself is a given curve. Therefore, the subtangent at the point B or E (because, just as is BE an infinitely small element, the subtangent is certainly the limit), answering $= T$; And calling $AE = f$, $BD = g$, $EB = m$, $DC = n$, the vertical element Bb, that is $Bb - Aa = c$; the vertical element Cc, that is $Cc - Bb = e$; $bc = \sqrt{gg + mm + ee}$. The element of the sought-for curve will be

$$ab = \sqrt{ff + mm + cc}, \text{ and } bc = \sqrt{gg + nn + ee}.$$

Therefore, the quantity

$$\sqrt{ff + mm + cc} + \sqrt{gg + mm + ee}$$

should be a minimum; differentiating therefore, (taking AE and BD, or f and g to be constants),

$$\frac{m\,dm + c\,dc}{\sqrt{ff + mm + cc}} + \frac{n\,dn + e\,de}{\sqrt{gg + mm + ee}} = 0.$$

Whence, because $m + n$, like $c + e$, are constants, thus

$$dn = -dn, \text{ and } de = -dc,$$

so that

$$\frac{m\,dm + c\,dc}{\sqrt{ff + nm + cc}} = \frac{n\,dm + e\,dc}{\sqrt{gg + na + ee}}.$$

However, in order that dm and dc can be eliminated, it is required of them the ratio, which be made thus: Saying that $Bb = z$, evidently $T\colon z = B\zeta$, or $\zeta\beta - Bb$ or dc; and indeed $dc = z\,dm\colon T$, because, substituting in the given equation, and dividing it by dm, and multiplying it by T, will produce

$$\frac{mT + cz}{\sqrt{ff + mm + cc}} = \frac{nT + ez}{\sqrt{gg + nn + cc}},$$

where not yet discovered is the uniform progression from the elements AE and DB to the elements BD, DC, because the common members of each equation are T and z; on account of which I arrange this in the manner

$$T \times \frac{n}{\sqrt{gg + mm + ee}} - \frac{m}{\sqrt{ff + mm + cc}} = z \times \left(\frac{-e}{\sqrt{gg + nn + ee}} + \frac{c}{\sqrt{ff + mm + cc}} \right).$$

Here, in the infinitely small factors is clearly observed a uniformity; for each denotes a uniform differential fraction of similar compound elements, so written n, g, e, these, which represent dy, dx, dz, we have

$$T \times d\left(\frac{dy}{\sqrt{dx^2 + dy^2 + dz^2}} \right) = z \times d\left(\frac{dz}{\sqrt{dx^2 + dy^2 + dz^2}} \right),$$

in which equation nothing constant is supposed, and therefore some element may be freely supposed invariable. Therefore we assume, as in *Klingenstierna's* Writing, the constant $\sqrt{dx^2 + dy^2}$, or ds, in order that the equation is expressed in this manner:

$$T \times d\left(\frac{dy}{\sqrt{ds^2 + dz^2}} \right) = z \times d\left(\frac{dz}{\sqrt{ds^2 + dz^2}} \right),$$

which fractions being differentiated, changes to

$$\frac{dz\,ddz}{ds^2 + dz^2} = \frac{T\,ddy + z\,ddx}{T\,dy + z\,dz},$$

equivalent to that given in the Additions to the previously cited writing, in order that the desired calculation is clearly seen. Because if in truth for constants are assumed AB itself, or the element of the desired curve, that is, $\sqrt{ds^2 + dz^2}$, an extremely simple equation emerges, certainly: $T\,ddy = -z\,ddz$. But nonetheless, this cannot generally be reduced to principal differences.